Functions

Part Two

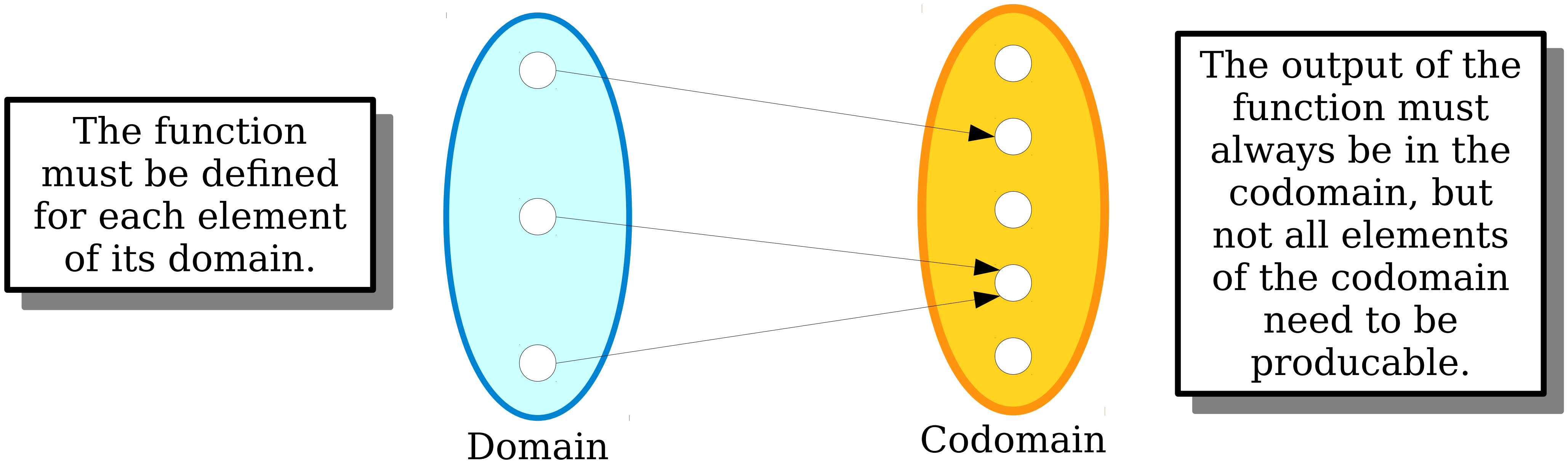
Outline for Today

- ***Recap from Last Time***
  - Where are we, again?
- ***A Proof About Birds***
  - Trust me, it's relevant. 😃
- ***Assuming vs Proving***
  - Two different roles to watch for.
- ***Connecting Function Types***
  - Relating the topics from last time.
- ***Function Composition***
  - Sequencing functions together.

Recap from Last Time

Domains and Codomains

- Every function *f* has two sets associated with it: its ***domain*** and its ***codomain***.
- A function *f* can only be applied to elements of its domain. For any *x* in the domain, *f*(*x*) belongs to the codomain.
- We write $\boldsymbol{f : A \to B}$ to indicate that *f* is a function whose domain is *A* and whose codomain is *B*.



Involutions

- A function $f : A \to A$ from a set back to itself is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

("Applying f twice is equivalent to not applying f at all.")

- For example, $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = -x$ is an involution.

Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) if different inputs always map to different outputs.
  - A function with this property is called an ***injection***.
- Formally, $f : A \to B$ is an injection if this FOL statement is true:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different"*)

- Equivalently:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same"*)

Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) if each element of the codomain is "covered" by at least one element of the domain.
  - A function with this property is called a ***surjection***.
- Formally, $f : A \to B$ is a surjection if this FOL statement is true:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every possible output, there's at least one possible input that produces it"*)

	To ***prove*** that this is true…	
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	
$A \to B$	Assume A is true, then prove B is true.	
$A \land B$	Prove A. Then prove B.	
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*	
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.	
$\neg A$	Simplify the negation, then consult this table on the result.	

New Stuff!

A Proof About Birds

Theorem: If all birds can fly,
then all herons can fly.

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Given the predicates

$Bird(b)$, which says b is a bird;
$Heron(h)$, which says h is a heron; and
$CanFly(x)$, which says x can fly,

translate the theorem into first-order logic.

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translate the theorem into first-order logic.

$$\underbrace{(\forall b.\ (Bird(b) \to CanFly(b)))}_{\text{All birds can fly}} \to \underbrace{(\forall h.\ (Heron(h) \to CanFly(h)))}_{\text{All herons can fly}}$$

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Consider an arbitrary bird *b*. Since *b* is a bird, *b* can fly. *[and now we're stuck! we are interested in herons, but b might not be one. It could be a hummingbird, for example!]*

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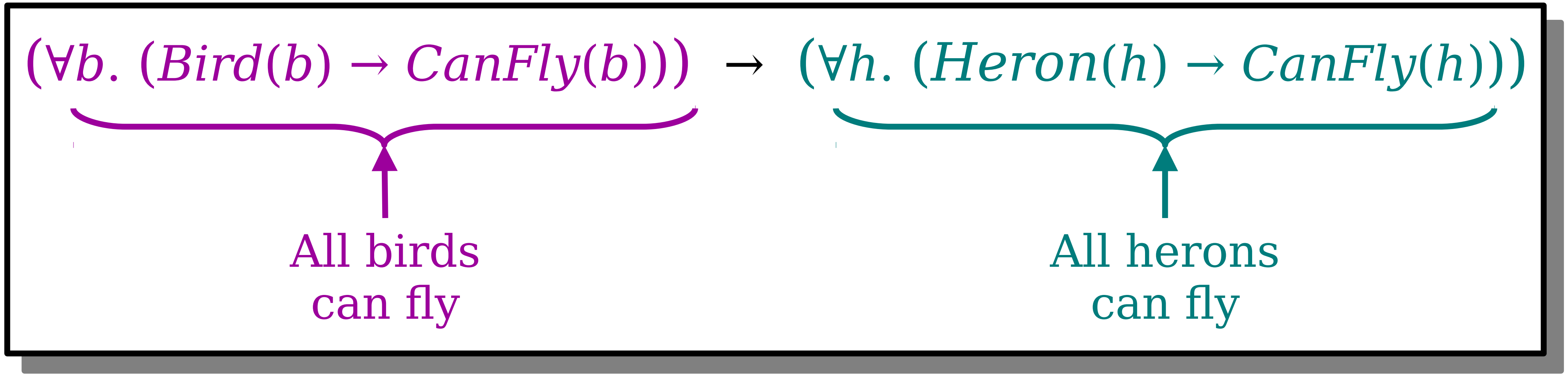

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We never introduce a variable b.

We introduce a variable h almost immediately.

Proving vs. Assuming

- In the context of a proof, you will need to assume some statements and prove others.
  - Here, we ***assumed*** all birds can fly.
  - Here, we ***proved*** all herons can fly.
- Statements behave differently based on whether you're assuming or proving them.

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Proving vs. Assuming

- To ***prove*** the universally-quantified statement

$$\forall x.\ P(x)$$

  we introduce a new variable x representing some arbitrarily-chosen value.
- Then, we prove that $P(x)$ is true for that variable x.
- That's why we introduced a variable h in this proof representing a heron.

$$(\forall b.\ (Bird(b) \to CanFly(b))) \to (\forall h.\ (Heron(h) \to CanFly(h)))$$

We never introduce a variable b.

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Proving vs. Assuming

- If we ***assume*** the statement

$$\forall x.\ P(x)$$

  we ***do not*** introduce a variable x.
- Rather, if we find a relevant value z somewhere else in the proof, we can conclude that $P(z)$ is true.
- That's why we didn't introduce a variable b in our proof, and why we concluded that h, our heron, can fly.

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$A \to B$	Assume A is true, then prove B is true.	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.
$A \land B$	Prove A. Then prove B.	Assume A. Then assume B.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*	Consider two cases. Case 1: A is true. Case 2: B is true.
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.	Assume $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Connecting Function Types

Types of Functions

- Last time, we saw three special types of functions:
  - ***involutions***, functions that undo themselves;
  - ***injections***, functions where different inputs go to different outputs; and
  - ***surjections***, functions that cover their whole codomain.
- ***Question:*** How do these three classes of functions relate to one another?

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

$$(\forall x \in A.\ f(f(x)) = x) \rightarrow (\forall b \in A.\ \exists a \in A.\ f(a) = b)$$

f is an involution.

f is surjective.

Theorem: For any function $f : A \rightarrow A$,
if *f* is an involution, then *f* is surjective.

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Assume this.

Prove this.

Since we're ***assuming*** this, we aren't going to pick a specific choice of *x* right now. Instead, we're going to keep an eye out for something to apply this fact to.

Proof Outline

1. Assume f is an involution.

Theorem: For any function $f : A \rightarrow A$, if f is an involution, then f is surjective.

$(\forall x \in A.\ f(f(x)) = x) \rightarrow (\forall b \in A.\ \exists a \in A.\ f(a) = b)$

Ass

We've said that we need to ***prove*** this statement. How do we do that?

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Now, we hit an existential quantifier. Since we're ***proving*** this, we need to find a choice of $a \in A$ where this is true.

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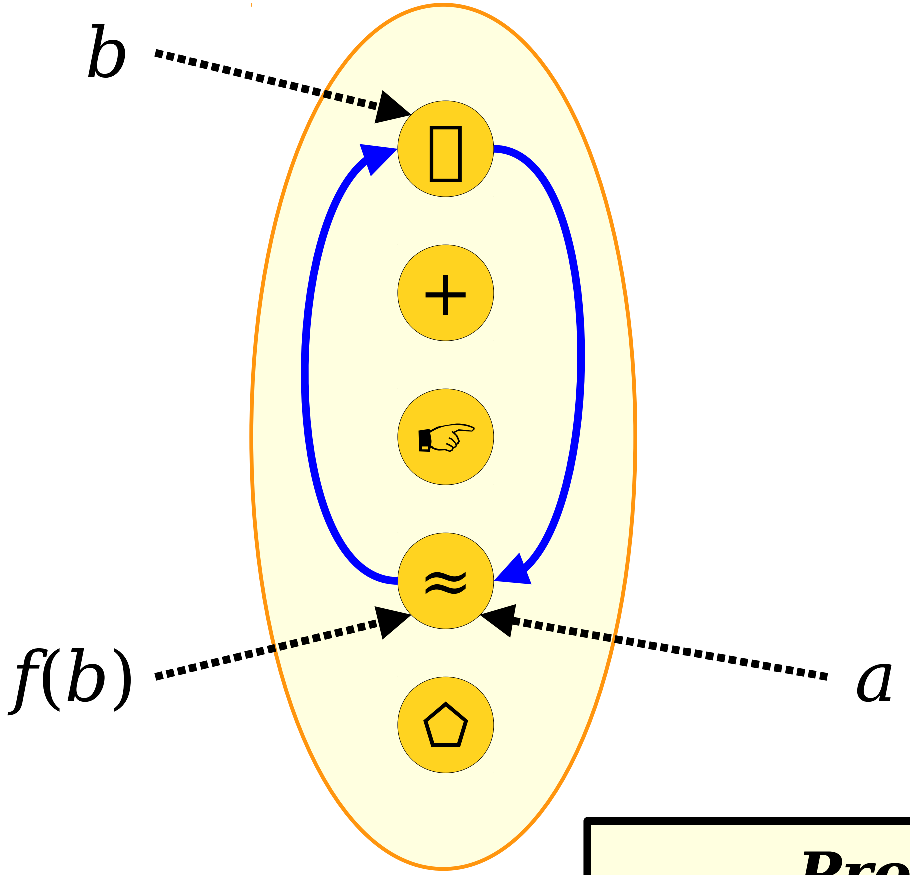

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Specifically, pick $a = f(b)$.

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Specifically, pick $a = f(b)$. This means that $f(a) = f(f(b))$, and since f is an involution we know that $f(f(b)) = b$. Putting this together, we see that $f(a) = b$, which is what we needed to show.

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Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

Proof: Pick any involution $f : A \to A$. We will prove that f is surjective. To do so, pick an arbitrary $b \in A$. We need to show that there is an $a \in A$ where $f(a) = b$.

Specifically, pick $a = f(b)$. This means that $f(a) = f(f(b))$, and since f is an involution we know that $f(f(b)) = b$. Putting this together, we see that $f(a) = b$, which is what we needed to show. ■

Proof Outline

1. Assume f is an involution.
2. Pick an arbitrary $b \in A$.
3. Give a choice of $a \in A$ where $f(a) = b$.

Theorem: For any function $f : A \to A$, if f is an involution, then f is injective.

$$\underbrace{(\forall x \in A.\ f(f(x)) = x)}_{f \text{ is an involution.}} \to \underbrace{(\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))}_{f \text{ is injective.}}$$

Theorem: For any function $f : A \to A$,
if f is an involution, then f is injective.

$$(\forall x \in A.\ f(f(x)) = x) \to (\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2)))$$

Assume this.

Prove this.

Theorem: For any function $f : A \to A$, if f is an involution, then f is injective.

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$$\underbrace{(\forall b.\ (Bird(b) \to CanFly(b)))}_{\text{Assume this.}} \to \underbrace{(\forall h.\ (Heron(h) \to CanFly(h)))}_{\text{Prove this.}}$$

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$(\forall x \in A.\ f(f(x)) = x) \rightarrow (\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \rightarrow f(a_1) \neq f(a_2)))$

Assume this.

Prove this.

Since we're ***assuming*** this, we aren't going to pick a specific choice of *x* right now. Instead, we're going to keep an eye out for something to apply this fact to.

Proof Outline

1. Assume *f* is an involution.

Theorem: For any function $f : A \rightarrow A$, if *f* is an involution, then *f* is injective.

$(\forall x \in A.\ f(f(x)) = x) \to (\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2)))$

We need to ***prove*** this part.
What does that mean?

Prove this.

Proof Outline

1. Assume f is an involution.

Theorem: For any function $f : A \to A$, if f is an involution, then f is injective.

$(\forall x \in A.\ f(f(x)) = x) \to (\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$

Since we're ***proving*** something universally-quantified, we'll pick some values arbitrarily.

Prove this.

Proof Outline

1. Assume f is an involution.

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Since we're ***proving*** something universally-quantified, we'll pick some values arbitrarily.

Prove this.

Proof Outline

1. Assume f is an involution.
2. Pick arbitrary a_1, $a_2 \in A$.

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We now need to ***prove*** this implication. But we know how to do that! We assume the antecedent and prove the consequent.

Prove this.

Proof Outline

1. Assume f is an involution.
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Prove this.

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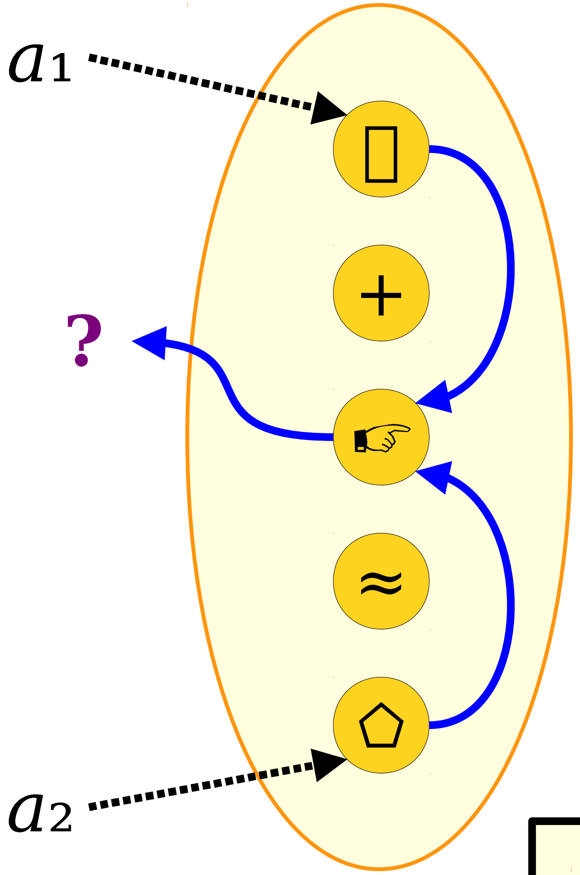

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We'll proceed by contradiction.

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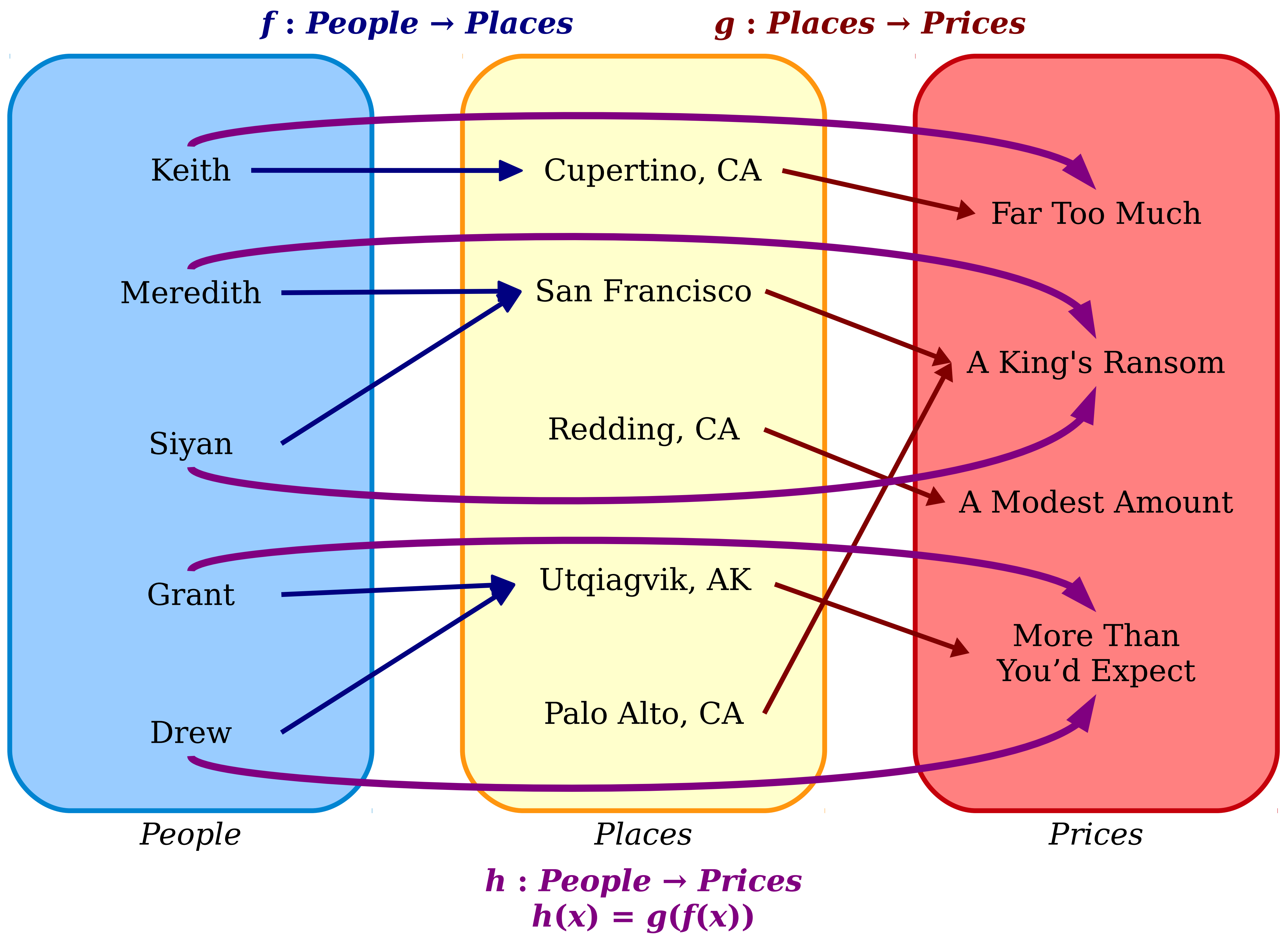
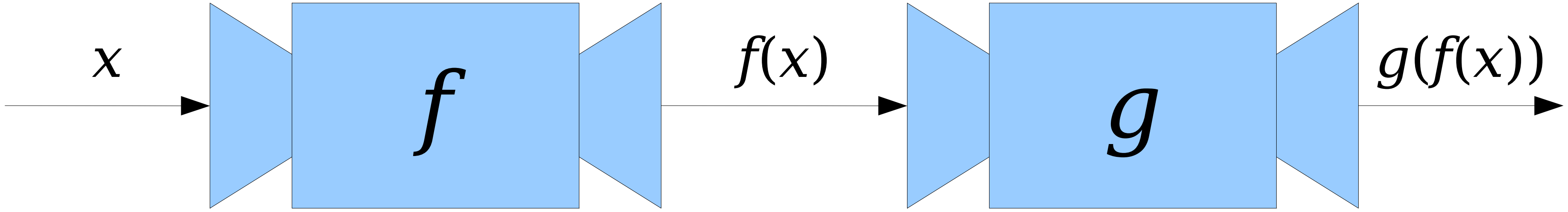
Function Composition



f : People → Places
g : Places → Prices
Keith
Meredith
Siyan
Grant
Drew
Cupertino, CA
San Francisco
Redding, CA
Utqiagvik, AK
Palo Alto, CA
Far Too Much
A King's Ransom
A Modest Amount
More Than
You’d Expect
People
Places
Prices
h : People → Prices
h(x) = g(f(x))

Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- Notice that the codomain of f is the domain of g. This means that we can use outputs from f as inputs to g.



Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- The ***composition of f and g***, denoted $\boldsymbol{g \circ f}$, is a function where
  - $g \circ f : A \to C$, and
  - $(g \circ f)(x) = g(f(x))$.

The name of the function is $g \circ f$. When we apply it to an input x, we write $(g \circ f)(x)$. I don't know why, but that's what we do.

- A few things to notice:
  - The domain of $g \circ f$ is the domain of f. Its codomain is the codomain of g.
  - Even though the composition is written $g \circ f$, when evaluating $(g \circ f)(x)$, the function f is evaluated first.

Properties of Composition

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

Organizing Our Thoughts

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming

$f : A \to B$ is an injection.

$\forall x \in A.\ \forall y \in A.\ (x \neq y \to$
$f(x) \neq f(y)$
$)$

$g : B \to C$ is an injection.

$\forall x \in B.\ \forall y \in B.\ (x \neq y \to$
$g(x) \neq g(y)$
$)$

We're ***assuming*** these universally-quantified statements, so we won't introduce any variables for what's here.

What We Need to Prove

$g \circ f$ is an injection.

$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to$
$(g \circ f)(a_1) \neq (g \circ f)(a_2)$
$)$

We need to ***prove*** this universally-quantified statement. So let's introduce arbitrarily-chosen values.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

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$a_1 \in A$ is arbitrarily-chosen.

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Now we're looking at an implication. Let's ***assume*** the antecedent and ***prove*** the consequent.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming	***What We Need to Prove***
$f : A \to B$ is an injection.	$g \circ f$ is an injection.
$\forall x \in A.\ \forall y \in A.\ (x \neq y \to f(x) \neq f(y))$	$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$
$g : B \to C$ is an injection.	Let's write this out separately and simplify things a bit.
$\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$	
$a_1 \in A$ is arbitrarily-chosen.	
$a_2 \in A$ is arbitrarily-chosen.	
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Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

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$f : A \to B$ is an injection.	$g \circ f$ is an injection.
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$f : A \to B$ is an injection.	$g \circ f$ is an injection.
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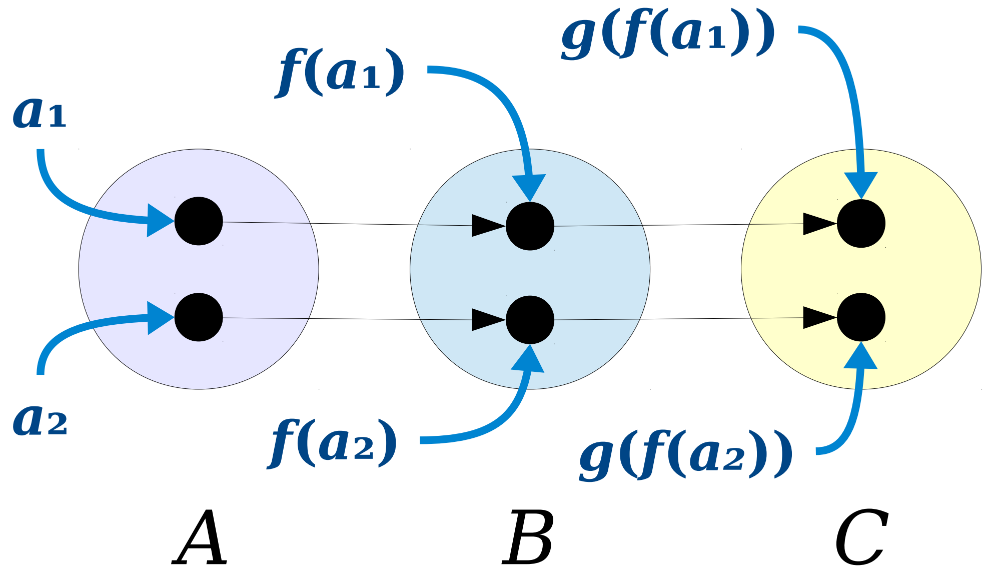
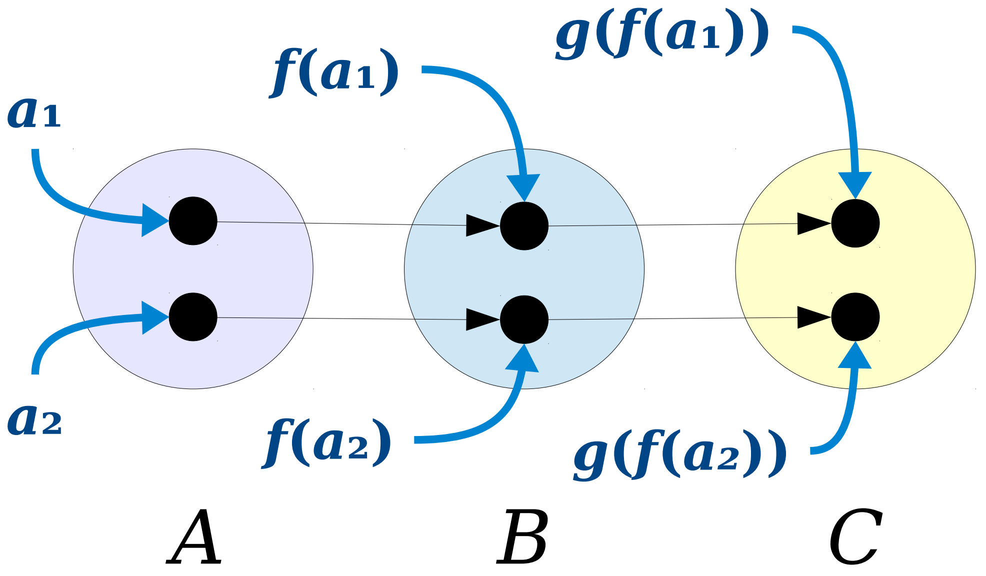
$g(f(a_1)) \neq g(f(a_2))$



Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.



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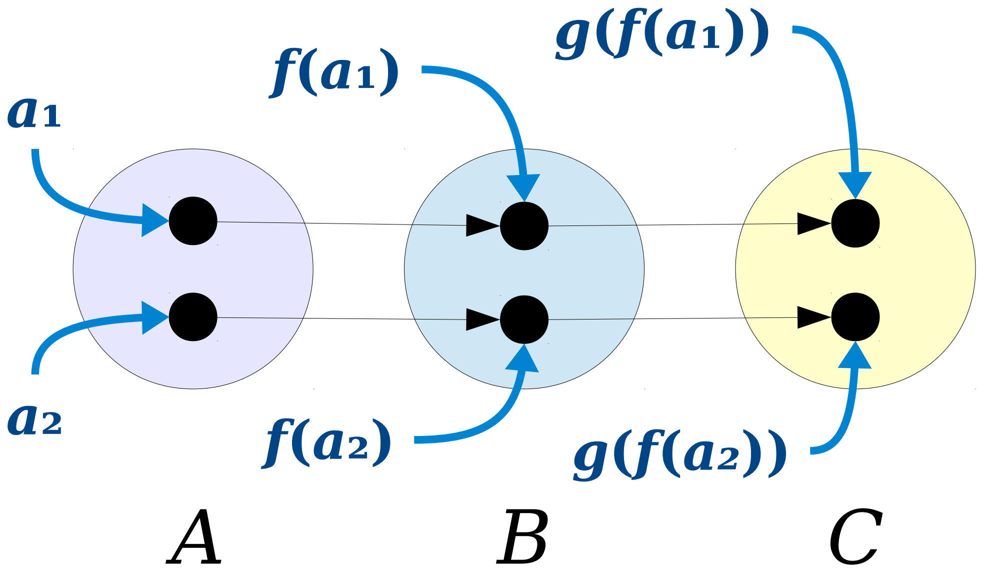
Proof:



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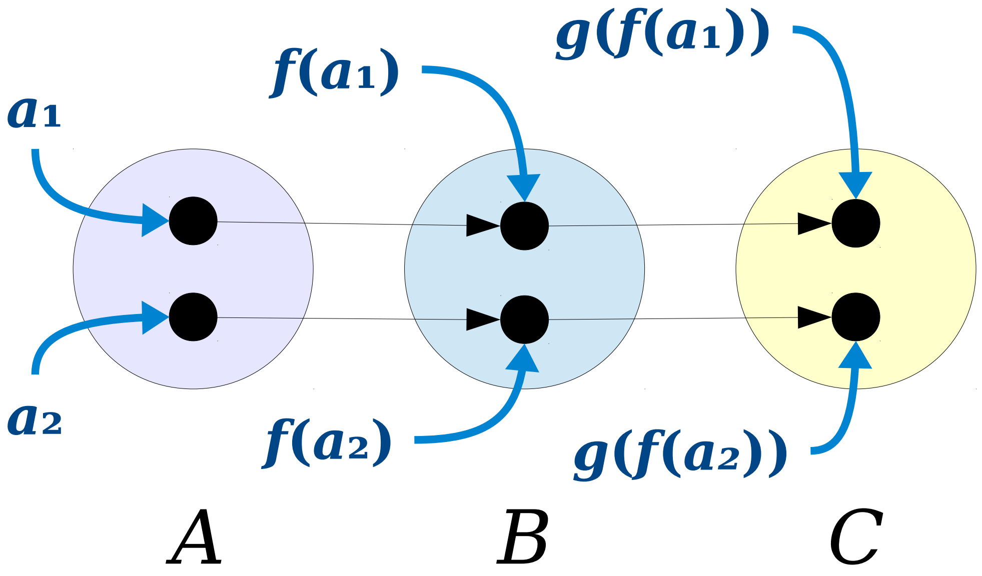
Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections.



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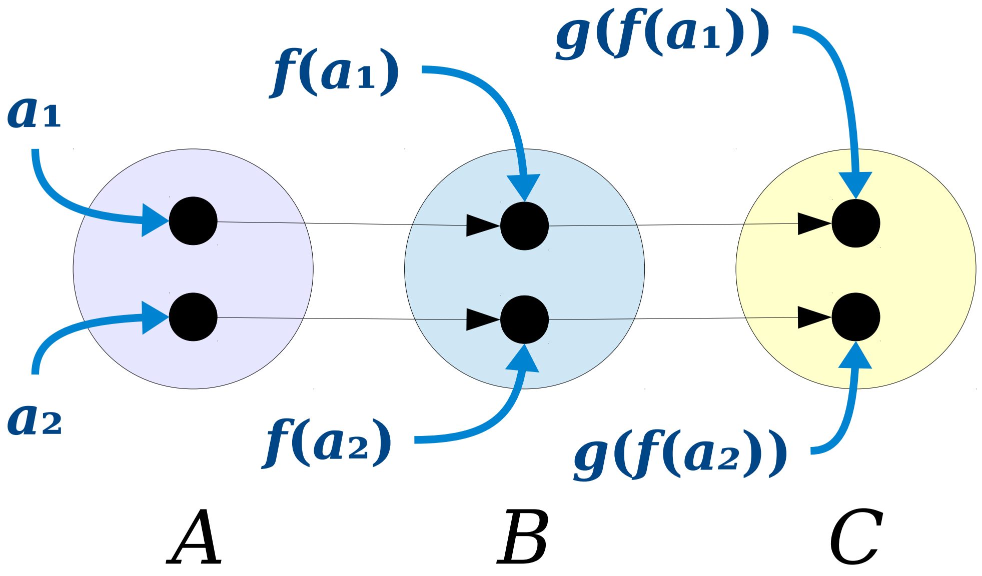
Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective.



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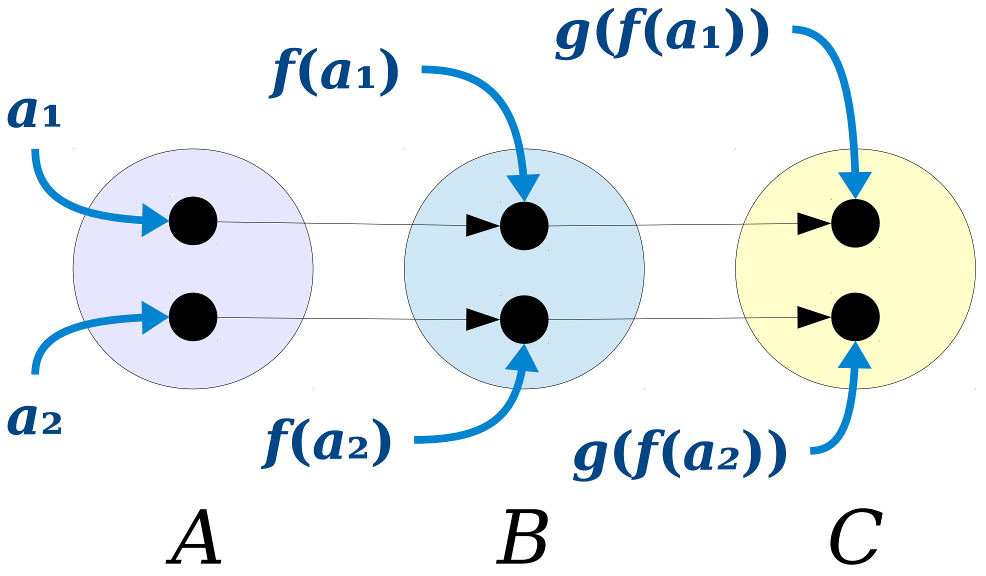

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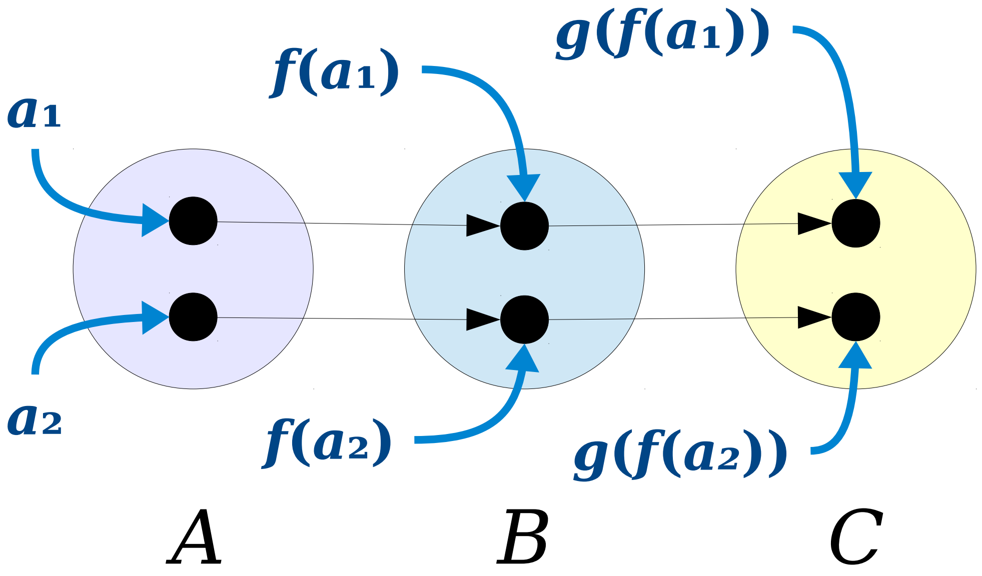

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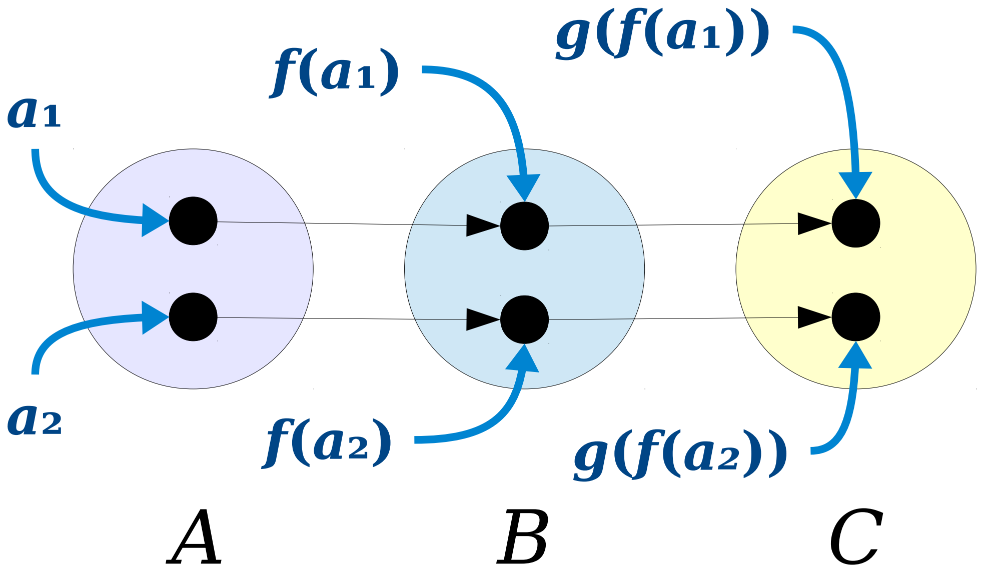

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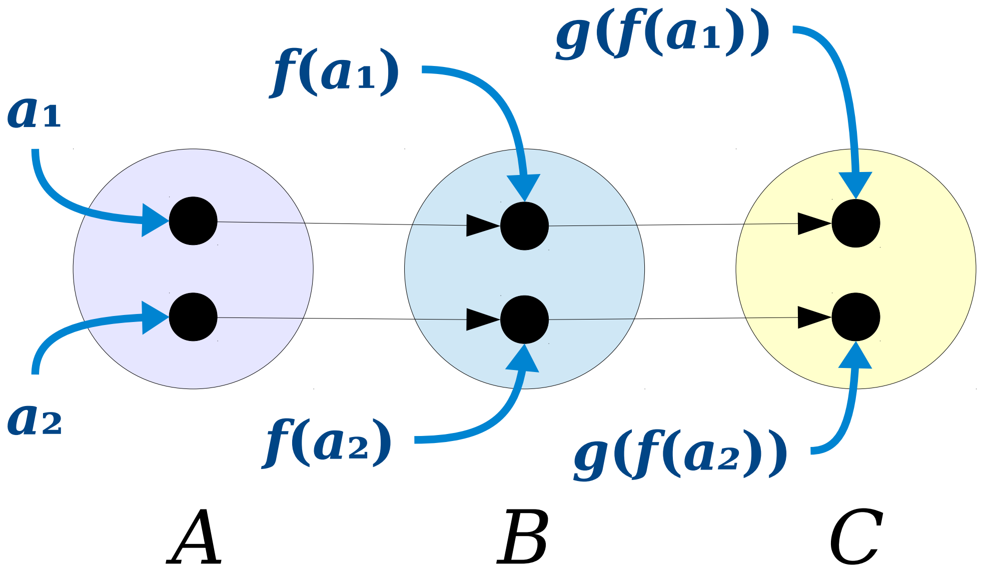

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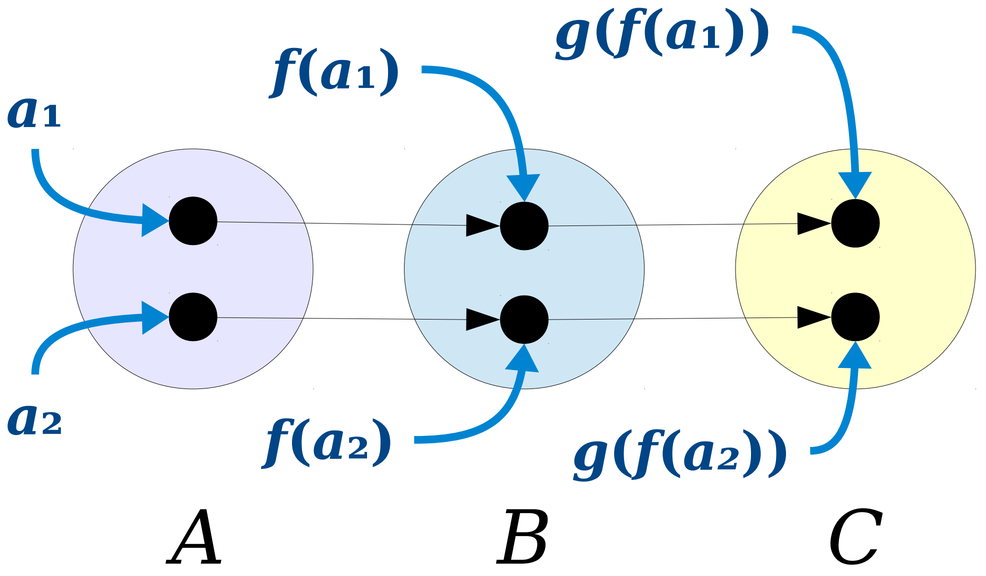

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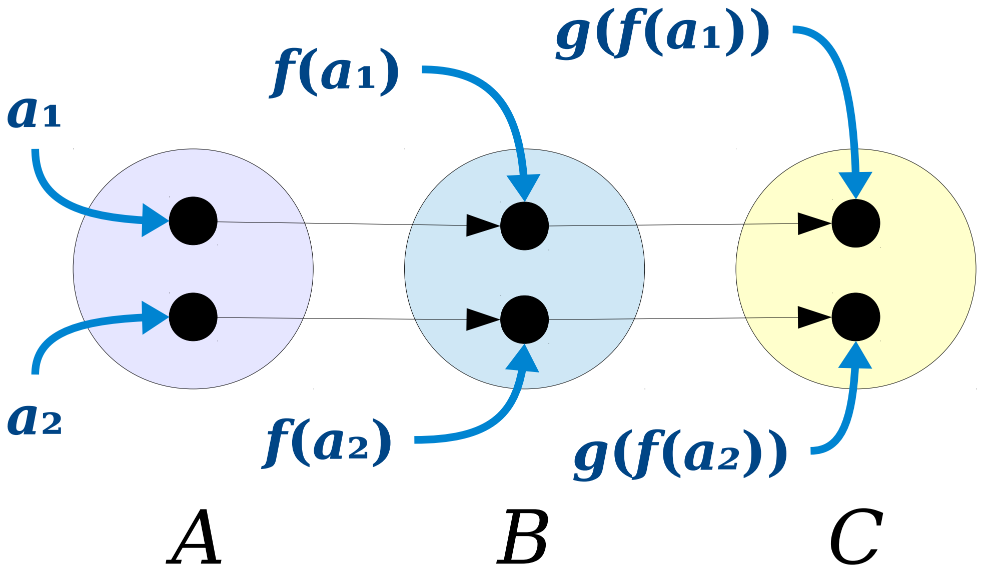

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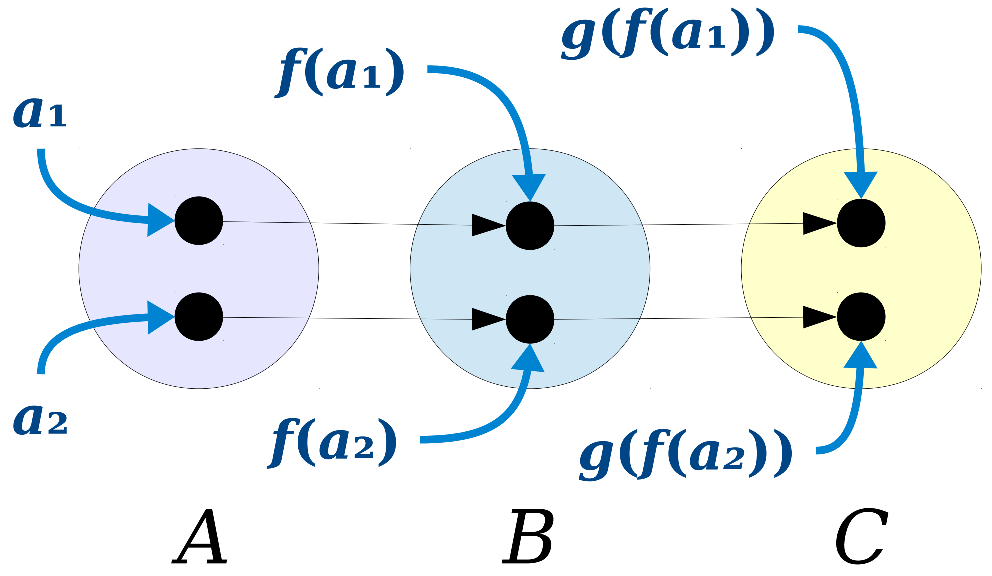
Great exercise: Repeat this proof using the other definition of injectivity.



Major Ideas From Today

- Statements behave differently based on whether you're ***assuming*** or ***proving*** them.
- When you ***assume*** a universally-quantified statement, initially, do nothing. Instead, keep an eye out for a place to apply the statement more specifically.
- When you ***prove*** a universally-quantified statement, pick an arbitrary value and try to prove it has the needed property.
- As always: try concrete examples, draw pictures, etc. before you dive into writing a proof.

	To ***prove*** that this is true…	If you ***assume*** this is true…
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	Introduce a variable x into your proof that has property A.
$A \to B$	Assume A is true, then prove B is true.	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.
$A \land B$	Prove A. Then prove B.	Assume A. Then assume B.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*	Consider two cases. Case 1: A is true. Case 2: B is true.
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.	Assume $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

Next Time

- ***Cardinality Revisited***
  - Formalizing our definitions.
- ***The Nature of Infinity***
  - Infinity is more interesting than it looks!
- ***Cantor's Theorem Revisited***
  - Formally proving a major result.

Extra Slides

(The following is a proof of a theorem just like the one we just did with injection, but with surjection.)

Theorem: If $f : A \to B$ is a surjection and $g : B \to C$ is a surjection, then the function $g \circ f : A \to C$ is a surjection.

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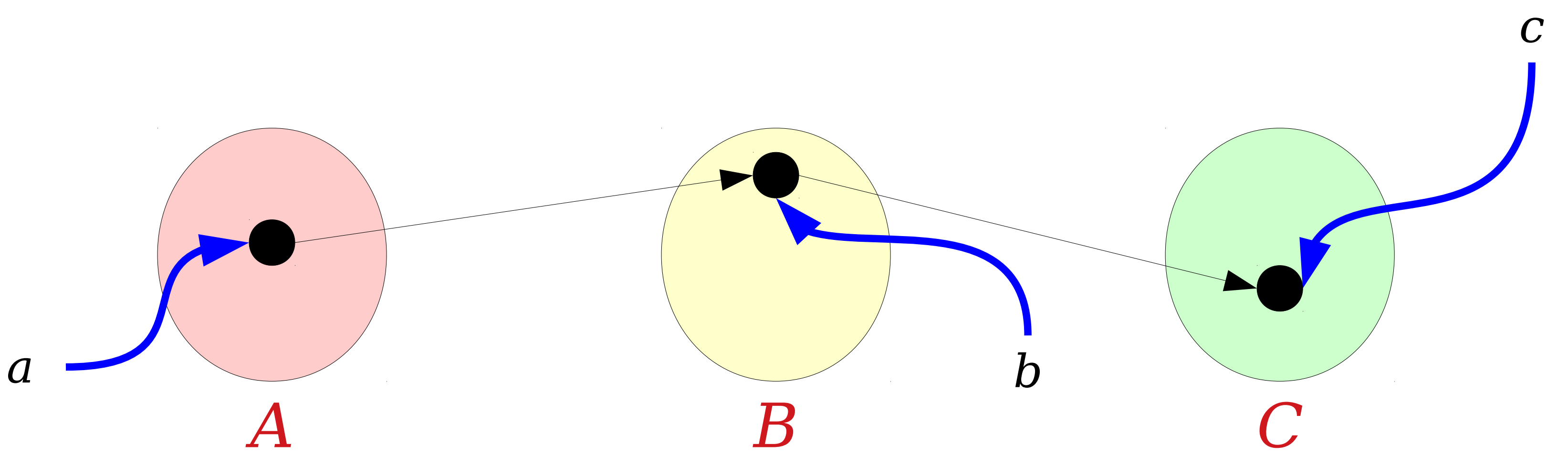

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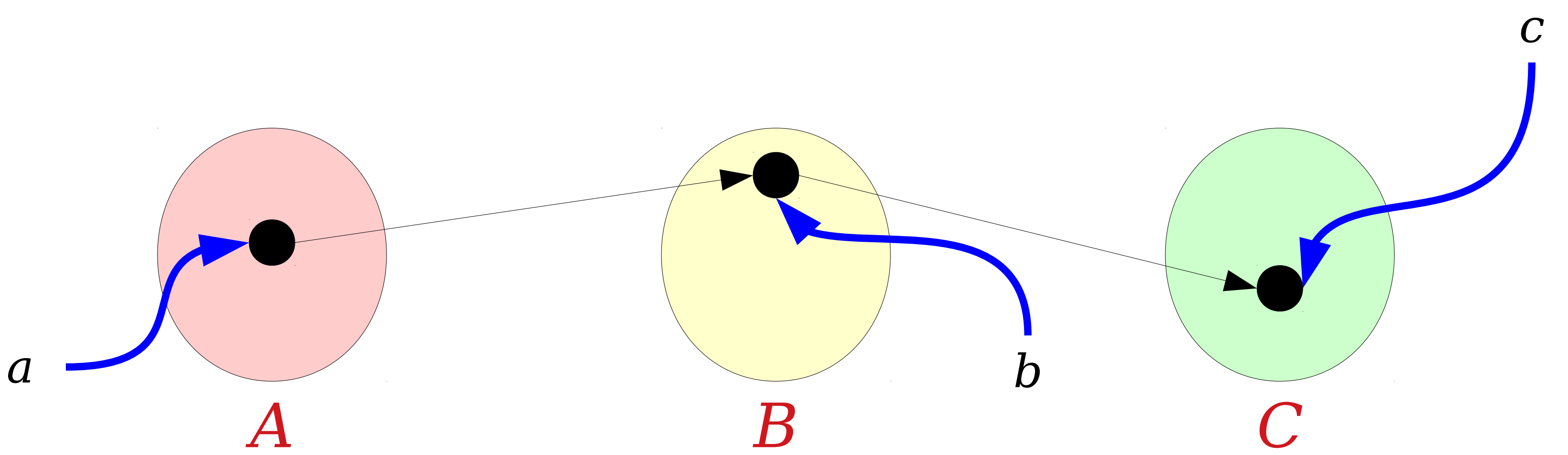

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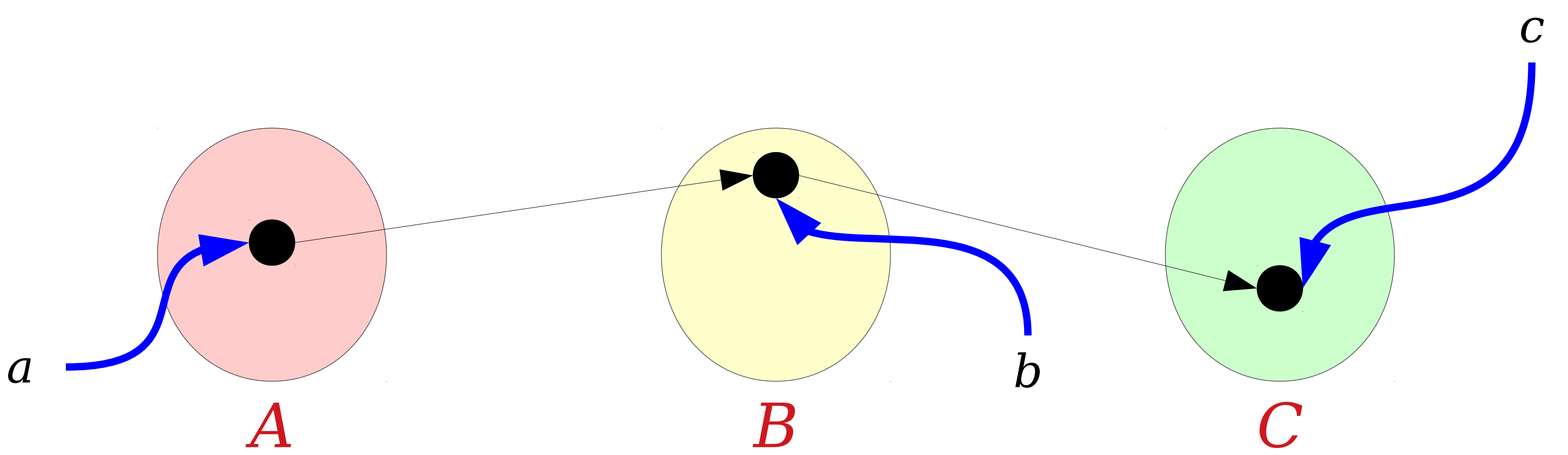

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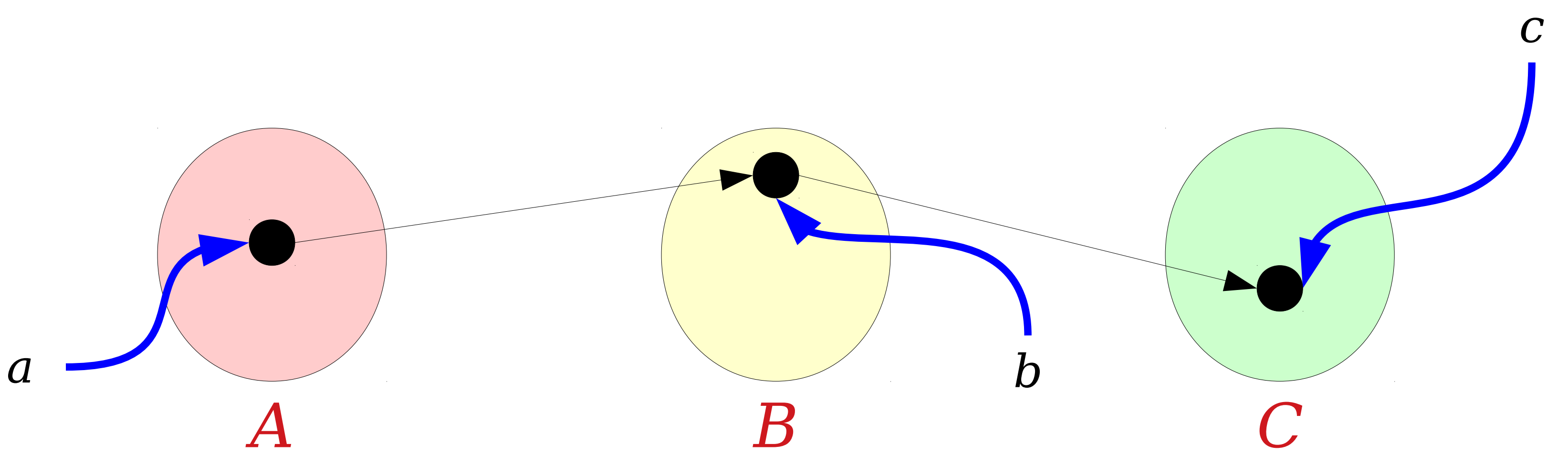

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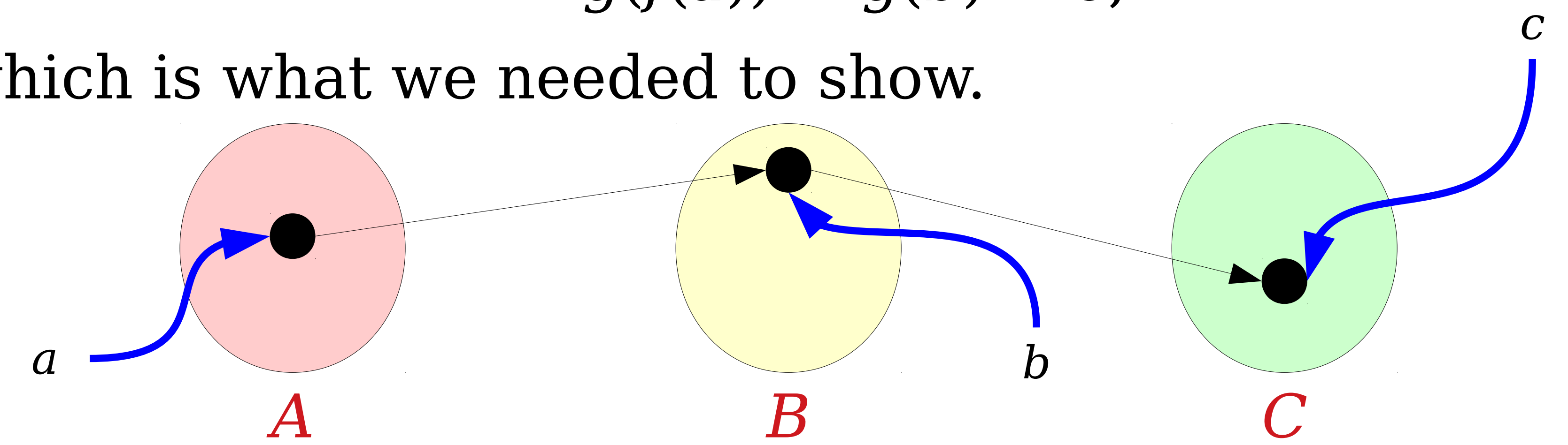

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